Complex Rational Numbers in Quantum mechanics

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February 1, 2008

Abstract

A binary representation of complex rational numbers and their arithmetic is described that is not based on qubits. It takes account of the fact that $0s$ in a qubit string do not contribute to the value of a number. They serve only as place holders. The representation is based on the distribution of four types of systems, corresponding to $+1, -1, +i, -i$, along an integer lattice. Complex rational numbers correspond to arbitrary products of four types of creation operators acting on the vacuum state. An occupation number representation is given for both bosons and fermions.

1 Introduction

Quantum computation and quantum information continue to attract much interest and study. Much of the interest was stimulated by work showing that quantum computers could solve some problems more efficiently than any known classical computer [1, 2]. Also some recent work addresses the possible relation between quantum computing and questions in cosmology and quantum gravity [3, 4].

In all of this work qubits (or qudits for d-dimesional systems) play a basic role. As quantum binary systems the states $|0\rangle, |1\rangle$ of a qubit represent the binary choices in quantum information theory. They also represent the numbers 0 and 1 as numerical inputs to quantum computers. For n qubits, corresponding product states, such as $|\underline{s}\rangle = \otimes_{j=1}^{n}|s(j)\rangle$ where $s(j) = 0$ or 1, represent a specific n qubit information state. These states and their linear superpositions are inputs to quantum computers.

Even though representation of numbers as strings of qubits or as strings of bits in classical work is widespread [5], it is not essential. Other representations are possible and may be useful. One of these is based on the observation that the $0s$ in a qubit or bit string serve only as place holders. They do not contribute to the value of the number.

This suggests a representation that does not use qubits or bits. It is based on the distributions of $1s$ along an integer lattice. For example the rational binary number 10100.0011 would be represented here as $1_4 1_2 1_{-3} 1_{-4}$.

This will be taken over to quantum mechanics by representing complex rational numbers as states of systems on a discrete lattice. The representation will be based on the use of annihilation and creation operators that create and annihilate systems on a lattice. Such an approach is useful in cases where particle numbers are not conserved. This is the case here as arithmetic operations do not conserve the number of $1s$ in numbers.[1]

Both bosons and fermions will be considered. For bosons the basic creation operators are $a^{\dagger}_{\alpha,j}$ and $b^{\dagger}_{\beta,k}$. The state $|0\rangle$ is the vacuum state. The single particle states $a^{\dagger}_{\alpha,j}|0\rangle$ and $b^{\dagger}_{\beta,k}|0\rangle$ show a type a system in internal state α at lattice site j and a type b system in internal state β at site k. They correspond respectively to the real and imaginary numbers $\alpha 2^j$ and $i\beta 2^k$ where $\alpha = +,-$ and $\beta = +,-$ denote the sign of the numbers. Multiple particle states and linear superpositions of these states are described using products of these operators, as in $1/\sqrt{2}(a^{\dagger}_{+,7}a^{\dagger}_{-,6}b^{\dagger}_{-,4}|0\rangle + a^{\dagger}_{-,-2}b^{\dagger}_{-,6}|0\rangle)$.

For fermions an additional discrete index $h = 1, 2\cdots$ is needed. This index does not contribute to the numerical value of a state but it does allow such states as $a^{\dagger}_{\alpha,j,1}a^{\dagger}_{\alpha,j,2}|0\rangle$ which corresponds to the number $\alpha(2^j + 2^j)$. Such states arise naturally during arithmetic operations.

This work differs from other work on fermionic quantum computation [6, 7] and number representation in quantum mechanics [8] in that it is not based on logical or physical qubits. It also differs in the use of an occupation number representation which results in both standard and nonstandard representations of complex rational numbers. These representations are defined in the next section.

The use of different a and b systems to represent real and imaginary numbers is done here to help understanding. It is completely arbitrary in that one can also use just one type of system with an extra internal state index as in $a^{\dagger}_{\alpha,x,j}$ (bosons) or $a^{\dagger}_{\alpha,x,j,h}$ (fermions) where $x = r, i$.

Here complex rational numbers are represented by all finite products of creation operators acting on the vacuum state $|0\rangle$. These states and their linear superpositions form a Fock space $\mathcal{H}^{Ra}$ of states. This representation is quite compact in that all four types of numbers can be included in one state. For example, the boson state $a^{\dagger}_{+,2}a^{\dagger}_{-,0}b^{\dagger}_{-,3}b^{\dagger}_{+,-1}a^{\dagger}_{-,-2}|0\rangle$, which represents the number $2^2 - 2^0 - 2^{-2} + i(-2^3 + 2^{-1})$, has the qubit representation $|10.11\rangle, |-i111.1\rangle$.

This flexibility and compactness allows a representation of many positive and negative complex rational numbers, which are to be combined into one complex number, as a single operator product acting on the vacuum. Such collections or matrices of numbers can occur, for example, in evaluating integrals of complex functions. Here one may collect many values of a function $f(x)$ over a range $u < v$ which are combined to evaluate the integral $\int_u^v f(x)dx$.

[1] They also do not conserve bit string lengths. This is usually accounted for by use of truncation to a fixed accuracy dependent length in computations.

In the next section basic properties of the annihilation creation (a-c) operators and their use in rational number states are outlined for both bosons and fermions. The most general states, which can have more than one system at a j lattice site, and their reductions to standard complex rational number states are discussed. Section 4 summarizes basic arithmetic operations on the states and their relation to complex number equivalents in C of rational numbers. Here C is the complex number field over which $\mathcal{H}^{Ra}$ is defined. More details on complex rational number states are given in [9].

It should be emphasized that rational number states and their arithmetic operations (addition, subtraction, multiplication and division to arbitrary accuracy) will be described here with no reference to the numbers in C and their arithmetic properties. An operator $\tilde{N}$ will be described that associates a complex rational number in C to each complex rational number state. The fact that $\tilde{N}$ is a morphism and preserves arithmetic properties is satisfying, but it plays no role in defining the properties of and operations on the states.

Finally one should note that the states described here as complex rational states do not correspond to all rational number states. However they are dense in the rational number states and can approximate any rational number state to arbitrary accuracy. For example, no rational state defined here corresponds to the number 1/3. However the rational states correspond to numbers that approximate 1/3 to arbitrary accuracy. They also correspond to the types of numbers used by computers in actual calculations.

2 Complex Rational Number States

One begins with the commutation relations for the basic a-c operators. For bosons one has

$$\begin{gathered}[a_{\alpha,j},a_{\alpha',k}]=[b_{\beta,j},b_{\beta',k}]=[a^{\dagger}_{\alpha,j},a^{\dagger}_{\alpha',k}]=[b^{\dagger}_{\beta,j},b^{\dagger}_{\beta',k}]=0\\ [a_{\alpha,j},a^{\dagger}_{\alpha',k}]=\delta_{\alpha,\alpha'}\delta_{j,k};\quad [b_{\beta,j},b^{\dagger}_{\beta',k}]=\delta_{\beta,\beta'}\delta_{j,k}.\end{gathered}\tag{1}$$

For fermions the anticommutation relations are

$$\begin{gathered}\{a_{\alpha,j,h},a_{\alpha',k,h'}\}=\{b_{\beta,j,h},b_{\beta',k,h'}\}=\{a^{\dagger}_{\alpha,j,h},a^{\dagger}_{\alpha',k,h'}\}=\{b^{\dagger}_{\beta,j,h},b^{\dagger}_{\beta',k,h'}\}=0\\ \{a_{\alpha,j,h},a^{\dagger}_{\alpha',k,h'}\}=\delta_{\alpha,\alpha'}\delta_{j,k}\delta_{h,h'};\quad \{b_{\beta,j,h},b^{\dagger}_{\beta',k,h'}\}=\delta_{\beta,\beta'}\delta_{j,k}\delta_{h,h'}\end{gathered}\tag{2}$$

where $\{c,d\}=cd+dc$. The a and b operators commute for both bosons and fermions as they represent distinguishable systems.

A complete basis set of states can be defined in terms of occupation numbers of the various boson or fermion states. Let n_+,n_-,m_+,m_- be any four functions that map the set of all integers to the nonnegative integers. Each function has the value 0 except possibly on a finite set of integers. Let s,s',t,t' be the four finite sets of integers which are the nonzero domains, respectively, of the four functions. Thus $n_{+,j}\neq 0[=0]$ if $j\epsilon s[j$ not in $s]$, $n_{-,j}\neq 0[=0]$ if $j\epsilon s'[j$ not in $s']$, etc.

Let $\bigcup s,t$ be the set of all integers in one or more of the four sets. Then a general boson occupation number state has the form

$$|n_+,n_-,m_+,m_-\rangle = \prod_{j\epsilon\cup s,t} |n_{+,j},n_{-,j}m_{+,j}m_{-,j}\rangle \tag{3}$$

where $|n_{+,j},n_{-,j}m_{+,j}m_{-,j}\rangle$, the occupation number state for site j, is given by

$$\begin{aligned}|n_{+,j},n_{-,j}m_{+,j}m_{-,j}\rangle &= \frac{1}{N(n,m,+,-,j)} \\ &\times(a^\dagger_{+,j})^{n_{+,j}}(a^\dagger_{-,j})^{n_{-,j}}(b^\dagger_{+,j})^{m_{+,j}}(b^\dagger_{-,j})^{m_{-,j}}|0\rangle.\end{aligned} \tag{4}$$

The normalization factor $N(n,m,+,-,j) = (n_{+,j}!n_{-,j}!m_{+,j}!m_{-,j}!)^{1/2}$. Note that the product $\prod_{j\epsilon\cup s,t}$ denotes a product of creation operators, and not a product of states.

The equivalent fermionic representation for the state $|n_+,n_-,m_+,m_-\rangle$ is based on a fixed ordering of the a-c operators. In this case the product $(a^\dagger_{+,j})^{n_{+,j}}$ becomes $a^\dagger_{+,1,j}\cdots a^\dagger_{+,h,j}\cdots a^\dagger_{+,n_{+,j},j}$ with similar replacements for $(a^\dagger_{-,j})^{n_{-,j}}$, $(b^\dagger_{+,j})^{m_{+,j}}$, and $(b^\dagger_{-,j})^{m_{-,j}}$. Each component state $|n_{+,j},n_{-,j},m_{+,j},m_{-,j}\rangle$ in Eq. 4 is given by

$$\begin{aligned}|n_{+,j},n_{-,j},m_{+,j},m_{-,j}\rangle &= a^\dagger_{+,n_{+,j},j}\cdots a^\dagger_{+,1,j}a^\dagger_{-,n_{-,j},j}\cdots \\ & a^\dagger_{-,1,j}b^\dagger_{+,m_{+,j},j}\cdots b^\dagger_{+,1,j}b^\dagger_{-,m_{-,j},j}\cdots b^\dagger_{-,1,j}|0\rangle\end{aligned} \tag{5}$$

The final state is given by an ordered product over the j values,

$$|n_+,n_-,m_+,m_-\rangle = \prod_{j\epsilon\cup s,t} J|n_{+,j},n_{-,j}m_{+,j}m_{-,j}\rangle. \tag{6}$$

Here J denotes a j ordered product where factors with larger values of j are to the right of factors with smaller j values. The choice of ordering, such as that used here in which the ordering of the j values is the opposite of that for the h values which increase to the left as in Eq. 5, is arbitrary. However, it must remain fixed throughout.

The interpretation of these states is that they are the boson or fermion equivalent of *nonstandard* representations of complex rational numbers as distinct from *standard* representations.[2] Such nonstandard states occur often in arithmetic operations. They correspond to columns of binary numbers where each number in the column is any one of the four types, positive real, negative real, positive imaginary, and negative imaginary. In a boson representation, individual systems are not distinguishable. The only measurable properties are the number of systems of each type $+1,-1,+i,-i$ in the single digit column at each site j. Individual systems are distinguishable in a fermion representation. However the variable h that separates fermions with the same value of α and j does not contribute to the numerical value of the state.

[2]This use of standard and nonstandard is completely different from standard and nonstandard numbers described in mathematical logic [10].

An already mentioned example of a nonstandard representation is that made by a computation of the value of the integral $\int_u^v f(x)dx$ of a complex valued function f. The table, or matrix, of M results obtained by computing in parallel, or by a quantum computation, values of $f(x_\ell)$ for $\ell = 1, 2, \cdots, M$ is represented here by a state $|n_+, n_-, m_+, m_-\rangle$ where $n_{+,j}, n_{-,j} m_{+,j}, m_{-,j}$ give the number of $+1's$, $-1's$, $+i's$, and $-i's$ in the column at site j. This is a nonstandard representation because it is numerically equal to the final result which is a standard representation consisting of one real and one imaginary rational number, often represented as a pair, w, iy.

Most of the occupation number states are nonstandard representations. The standard representations are characterized by the restrictions that at most one of n_+, n_- and one of m_+, m_- have nonempty domains and that the functions have the constant value 1 on their domains. The four possibilities are

$$\begin{aligned} |\underline{1}_s, 0, \underline{1}_t, 0\rangle &= (a_+^\dagger)^s (b_+^\dagger)^t |0\rangle \\ |\underline{1}_s, 0, 0, \underline{1}_{t'}\rangle &= (a_+^\dagger)^s (b_-^\dagger)^{t'} |0\rangle \\ |0, \underline{1}_{s'}, \underline{1}_t, 0\rangle &= (a_-^\dagger)^{s'} (b_+^\dagger)^t |0\rangle \\ |0, \underline{1}_{s'}, 0, \underline{1}_{t'}\rangle &= (a_-^\dagger)^{s'} (b_-^\dagger)^{t'} |0\rangle . \end{aligned} \tag{7}$$

Here $(a_+^\dagger)^s = \prod_{j \epsilon s} a_{+,j}^\dagger$ and $\underline{1}_s$ denotes the constant 1 function on s, etc. Pure real or imaginary standard rational states are included if $t,\ t'$ or $s,\ s'$ are empty. If $s,\ s',\ t,\ t'$ are all empty one has the vacuum state $|0\rangle$. Note that Eq. 7 is also valid for fermions with the replacements

$$\begin{aligned} (a_\alpha^\dagger)^s &\to a_{\alpha,1,j_1}^\dagger a_{\alpha,1,j_2}^\dagger \cdots a_{\alpha,1,j_{|s|}}^\dagger \\ (b_\beta^\dagger)^t &\to b_{\beta,1,k_1}^\dagger b_{\beta,1,k_2}^\dagger \cdots b_{\beta,1,k_{|t|}}^\dagger . \end{aligned} \tag{8}$$

Here $\alpha = +, -$, $\beta = +, -$, and $s = \{j_1, j_2, \cdots, j_{|s|}\}$, $t = \{k_1, k_2, \cdots, k_{|t|}\}$. Also $j_1 < j_2 < \cdots < j_{|s|}$, $k_1 < k_2 < \cdots < k_{|t|}$, and $|s|, |t|$ denote the number of integers in s, t.

Standard states are quite important. All theoretical predictions as computational outputs, and numerical experimental results are represented by standard real rational states. Nonstandard representations occur during the computation process and in any situation where a large amount of numbers is to be combined. Also qubit states correspond to standard representations only.

This shows that it is important to describe the numerical relations between nonstandard representations and standard representations and to define numerical equality between states. To this end let

$$|n_+, n_-, m_+, m_-\rangle =_N |n'_+, n'_-, m'_+, m'_-\rangle \tag{9}$$

be the statement that the two indicated states are numerically equal. Note that numerical equality has nothing to do with state equality in quantum mechanics. Two numerically equal states can be quite different physically.

Numerical equality is defined by some basic requirements on a-c operators. For bosons they are

$$a_{+,j}^\dagger a_{-,j}^\dagger =_N \tilde{1}; \quad b_{+,j}^\dagger b_{-,j}^\dagger =_N \tilde{1} \tag{10}$$

and

$$\begin{array}{ll} a^{\dagger}_{\alpha,j}a^{\dagger}_{\alpha,j} =_N a^{\dagger}_{\alpha,j+1} & a_{\alpha,j}a_{\alpha,j} =_N a_{\alpha,j+1} \\ b^{\dagger}_{\beta,j}b^{\dagger}_{\beta,j} =_N b^{\dagger}_{\beta,j+1} & b_{\beta,j}b_{\beta,j} =_N b_{\beta,j+1}. \end{array} \tag{11}$$

For fermions one has

$$a^{\dagger}_{+,j,h}a^{\dagger}_{-,j,h'} =_N \tilde{1}; \quad b^{\dagger}_{+,j,h}b^{\dagger}_{-,j,h'} =_N \tilde{1} \tag{12}$$

and

$$\begin{array}{ll} a^{\dagger}_{\alpha,h,j}a^{\dagger}_{\alpha,h',j} =_N a^{\dagger}_{\alpha,h'',j+1} & a_{\alpha,h,j}a_{\alpha,h',j} =_N a_{\alpha,h,'',j+1} \\ b^{\dagger}_{\beta,h,j}b^{\dagger}_{\beta,h',j} =_N b^{\dagger}_{\beta,h'',j+1} & b_{\beta,h,j}b_{\beta,h',j} =_N b_{\beta,h'',j+1}. \end{array} \tag{13}$$

In Eq. 13 $h \neq h'$. Otherwise the values of $h, h', h'' \geq 1$ are arbitrary except that removal of fermions is restricted to occupied h values and addition is restricted to unoccupied values. To avoid poking holes in the successive values of h at each site j, it is useful to restrict system removal to the maximum occupied h value and addition to its nearest unoccupied neighbor. However the h values at which systems are added or removed do not affect the numerical value of the state.

The first pair of equations says that any state that has one or more $+$ and $-$ systems of either the r (real) or i (imaginary) type at a site j is numerically equivalent to the state with one less $+$ and $-$ system at the site j of either type. This is the expression here of $2^j - 2^j = i2^j - i2^j = 0$ for the numbers in C. The second set of two pairs, Eq. 11, says that any state with two systems of the same type and in the same internal state at site j, and two different h values for fermions, is numerically equivalent to a state without these systems but with one system of the same type and internal state at site $j+1$. This corresponds to $2^j + 2^j = 2^{j+1}$ or $i2^j + i2^j = i2^{j+1}$.

From these relations one sees that any process whose iteration preserves N equality according to Eqs. 10 and 11 can be used to determine if Eq. 9 is valid for two different states. For example, for bosons if

$$|n_+, n_-, m_+, m_-\rangle = a_{+,j}a_{-,j}|n'_+, n'_-, m'_+, m'_-\rangle \tag{14}$$

or

$$|n_+, n_-, m_+, m_-\rangle = b^{\dagger}_{+,j+1}b_{+,j}b_{+,j}|n'_+, n'_-, m'_+, m'_-\rangle, \tag{15}$$

then Eq. 9 is satisfied.

One can use the a-c operators to define operators that carry out the changes on states implied by Eqs. 10, 11, 12, and 13. Explicit expressions are given in reference[9].

Reduction of a nonstandard representation to a standard one proceeds by iteration of steps based on the above equivalences. At some point the process stops when the resulting state has at most one system of the a or b type at each site j. This is the case for both bosons and fermions. The possible options for

each j can be expressed as

$$|n_{+,j}, n_{-,j}, m_{+,j}, m_{-,j}\rangle = \left\{ \begin{array}{l} |1,0,0,1\rangle \\ |1,0,1,0\rangle \\ |0,1,0,1\rangle \\ |0,1,1,0\rangle \end{array} \right. \text{ or } \left\{ \begin{array}{l} |0,0,0,1\rangle \\ |0,0,1,0\rangle \\ |1,0,0,0\rangle \\ |0,1,0,0\rangle \end{array} \right. \tag{16}$$

$$\text{or } |0,0,0,0\rangle.$$

An example of such a state for several j is $|1_{+,3}i_{+,3}1_{-,2}i_{-,4}1_{-,-6}\rangle$. This state corresponds to the C number $2^3 - 2^2 - 2^{-6} + i(2^3 - 2^4)$.

Conversion of a state in this form into a standard state requires first determining the signs of the a and b systems occupying the sites with the largest j values. This determines the signs separately for the real and imaginary components of the standard representation. In the example given above the real component is $+$ as $3 > 2, -6$ and the imaginary component is $-$ as $4 > 3$.

Conversion of all a-c operators into the same kind, as shown in Eq. 7, is based on four relations obtained by iteration of Eq. 11 and use of Eq. 10. For $k < j$ and for bosons they are

$$\begin{array}{l} a^\dagger_{+,j} a^\dagger_{-,k} =_N a^\dagger_{+,j-1} \cdots a^\dagger_{+,k} \\ a^\dagger_{-,j} a^\dagger_{+,k} =_N a^\dagger_{-,j-1} \cdots a^\dagger_{-,k} \\ b^\dagger_{+,j} b^\dagger_{-,k} =_N b^\dagger_{+,j-1} \cdots b^\dagger_{+,k} \\ b^\dagger_{-,j} b^\dagger_{+,k} =_N b^\dagger_{-,j-1} \cdots b^\dagger_{-,k}. \end{array} \tag{17}$$

These equations are used to convert all a and all b operators to the same type ($+$ or $-$) as the one at the largest occupied j value. Applied to the example $|1_{+,3}i_{+,3}1_{-,2}i_{-,4}1_{-,-6}\rangle$, gives $|1_{+,2}1_{+,1}1_{+,0}1_{+,-1}1_{+,-3} \cdots 1_{+,-6}i_{-,3}\rangle$. for the standard representation.

The same four equations hold for fermions provided h subscripts are included. The values of h are arbitrary as they do not affect $=_N$. However, physically, application to a state of the form of Eq. 16 requires that $h = 1$ everywhere, as in $a^\dagger_{+,1,j} a^\dagger_{-,1,k} =_N a^\dagger_{+,1,j-1} \cdots a^\dagger_{+,1,k}$ for example.

3 A Number Operator

It is useful to define an operator $\tilde{N}$ that assigns to each complex rational state a corresponding complex rational number in C. Each standard and nonstandard complex rational state is an eigenstate of $\tilde{N}$. The eigenvalue for this state is the complex number in C that $\tilde{N}$ associates with the state.

For fermions $\tilde{N}$ is defined by

$$\begin{array}{l} \tilde{N} = \sum_{h,j} 2^j [a^\dagger_{+,h,j} a_{+,h,j} - a^\dagger_{-,h,j} a_{-,h,j} \\ \quad + i(b^\dagger_{+,h,j} b_{+,h,j} - b^\dagger_{-,h,j} b_{-,h,j})]. \end{array} \tag{18}$$

From this definition one can obtain the following properties:

$$\begin{array}{c}[\tilde{N}, a^{\dagger}_{\alpha,h,j}] = \alpha 2^j a^{\dagger}_{\alpha,h,j} \\ [\tilde{N}, b^{\dagger}_{\beta,h,j}] = i\beta 2^j b^{\dagger}_{\beta,h,j} \\ \tilde{N}|0\rangle = 0.\end{array} \tag{19}$$

Here $\alpha = +, -$ and $\beta = +, -$. These equations apply to bosons if the h variable is deleted.

The eigenvalues of $\tilde{N}$ acting on states that are products of $a^{\dagger}$ and $b^{\dagger}$ operators can be obtained from Eqs. 18 or 19. As an example, for the boson state $a^{\dagger}_{+,k_1} a^{\dagger}_{-,k_2} b^{\dagger}_{+,k_3} b^{\dagger}_{-,k_4}|0\rangle$,

$$\begin{array}{c}\tilde{N} a^{\dagger}_{+,k_1} a^{\dagger}_{-,k_2} b^{\dagger}_{+,k_3} b^{\dagger}_{-,k_4}|0\rangle = \\ (2^{k_1} - 2^{k_2} + i2^{k_3} - i2^{k_4}) \\ \times a^{\dagger}_{+,k_1} a^{\dagger}_{-,k_2} b^{\dagger}_{+,k_3} b^{\dagger}_{-,k_4}|0\rangle.\end{array} \tag{20}$$

For standard representations in general

$$\tilde{N}(a^{\dagger}_{\alpha})^s (b^{\dagger}_{\beta})^t|0\rangle = N[(a^{\dagger}_{\alpha})^s (b^{\dagger}_{\beta})^t](a^{\dagger}_{\alpha})^s (b^{\dagger}_{\beta})^t|0\rangle \tag{21}$$

where

$$\tilde{N}[(a^{\dagger}_{\alpha})^s (b^{\dagger}_{\beta})^t] = \begin{cases} 2^s + i2^t & \text{if } \alpha = +, \beta = + \\ -2^s + i2^t & \text{if } \alpha = -, \beta = + \\ 2^s - i2^t & \text{if } \alpha = +, \beta = - \\ -2^s - i2^t & \text{if } \alpha = -, \beta = -. \end{cases} \tag{22}$$

Here $2^s = \sum_{j\epsilon s} 2^j$ and $2^t = \sum_{k\epsilon t} 2^j$.

These results also hold for fermion states. For standard states Eq. 8 gives an explicit representation for $(a^{\dagger}_{\alpha})^s (b^{\dagger}_{\beta})^t (a^{\dagger}_{\alpha})^s (b^{\dagger}_{\beta})^t|0\rangle$.

The operator $\tilde{N}$ has the satisfying property that any two states that are N equal have the same $\tilde{N}$ eigenvalue. If the state $|n_+, n_-, m_+, m_-\rangle =_N (a^{\dagger}_{\alpha})^s (b^{\dagger}_{\beta})^t|0\rangle$ then

$$\tilde{N}|n_+, n_-, m_+, m_-\rangle =_N \tilde{N}(a^{\dagger}_{\alpha})^s (b^{\dagger}_{\beta})^t|0\rangle. \tag{23}$$

This follows from Eqs. 10,11, and 18.

These results show that the eigenspaces of $\tilde{N}$ are invariant for any process of reducing a nonstandard state to a standard state using Eqs. 10-13. Any state $|n_+, n_-, m_+, m_-\rangle$ with $n_{+,j} \geq 1$ and $n_{-,j} \geq 1$ for some j has the same $\tilde{N}$ eigenvalue as the state with both $n_{+,j}$ and $n_{-,j}$ replaced by $n_{+,j}-1$ and $n_{-,j}-1$. Also if $n_{+,j} \geq 2$ then replacing $n_{+,j}$ by $n_{+,j}-2$ and $n_{+,j+1}$ by $n_{+,j+1}+1$ does not change the $\tilde{N}$ eigenvalue. Similar relations hold for $m_{-,j}, m_{+,j}, m_{-,j}$. These results show that each eigenspace of $\tilde{N}$ is infinite dimensional. It is spanned by an infinite number of nonstandard complex rational states and exactly one standard state.

The usefulness of $\tilde{N}$ results from the fact that it is a morphism from the complex rational number basis in $\mathcal{H}^{Ra}$ to the complex rational numbers in C. That is, it preserves arithmetic relations and operations. It was also used implicitly in the preceding to supply numerical values to states as illustrations.

It is important to note that $\tilde{N}$ is not used in any way to define the standard and nonstandard complex rational states or the basic properties of $=_N$. It will also not be used in the next section to define and give properties of basic arithmetic operations. The definitions and arithmetic properties of the complex rational states stand on their own with no reference to $\tilde{N}$. However, the operator can be used as a check to show that the arithmetic properties of the states are preserved by their $\tilde{N}$ images in C.

4 Arithmetic Operations

Here the definition and properties of arithmetic operations are limited to addition and subtraction. Also the discussion is limited to standard states and their linear superpositions. Extension to nonstandard states is straightforward as any state N equal to a standard state has the same arithmetic properties as the standard state. Details on nonstandard states and multiplication and division to any finite accuracy are given in reference[9].

It is useful to introduce a compact notation for standard states:

$$|\alpha s, \beta t\rangle = (a^{\dagger}_{\alpha})^{s}(b^{\dagger}_{\beta})^{t}|0\rangle. \tag{24}$$

A unitary addition operator, $\tilde{+}$, is defined by

$$\begin{aligned} &\tilde{+}|\alpha s, \beta t\rangle|\alpha' s', \beta' t'\rangle|0\rangle = \\ &\qquad |\alpha s, \beta t\rangle|\alpha' s', \beta' t'\rangle|\alpha s, \beta t + \alpha' s', \beta' t'\rangle \end{aligned} \tag{25}$$

where

$$\begin{aligned} &|\alpha s, \beta t + \alpha' s', \beta' t'\rangle \\ &\quad = (a^{\dagger}_{\alpha})^{s}(b^{\dagger}_{\beta})^{t}(a^{\dagger}_{\alpha'})^{s'}(b^{\dagger}_{\beta'})^{t'}|0\rangle = \\ &\quad\quad = (a^{\dagger}_{\alpha})^{s}(a^{\dagger}_{\alpha'})^{s'}(b^{\dagger}_{\beta})^{t}(b^{\dagger}_{\beta'})^{t'}|0\rangle \\ &\quad\quad\quad = |\alpha s + \alpha' s', \beta t + \beta' t'\rangle. \end{aligned} \tag{26}$$

This result, which uses the commutativity of the a and b a-c operators, shows the separate addition of the a and b components of the states.[3]

This also shows that the result of addition need not be a standard representation even if the inputs are standard. This is the case if s and s' or t and t' have common elements or if $\alpha \neq \alpha'$ or $\beta \neq \beta'$.

The notation of Eq. 24 will be used for both fermions and bosons with the understanding that for fermions the real component $(a^{\dagger}_{\alpha})^{s}(a^{\dagger}_{\alpha'})^{s'}$ is given by Eq. 8. Also if $\alpha = \alpha'$ then for any sites j that s, s' have in common, the operator product $a^{\dagger}_{\alpha,1,j}a^{\dagger}_{\alpha,1,j}$ is replaced by $a^{\dagger}_{\alpha,2,j}a^{\dagger}_{\alpha,1,j}$. Also for fermions the equality sign in Eq. 25 is replaced by $=_{\pm}$ or equality up to the sign. If the number of fermions in $|\alpha s, \beta t + \alpha' s', \beta' t'\rangle$ is odd the sign is minus. Otherwise it

[3]The product state representation is used here as it is familiar. The three states can be represented in the a-c operator formalism as a single state by expanding $\mathcal{H}^{Ra}$ to include operators for three distinguishable pairs of distinguishable systems (i.e. $(a, b) \rightarrow (a, b), (c, d), (e, f)$ or by adding an additional integral index to a, b that has the values $1, 2, 3$ for the three different states in the product. In this case there are just two distinguishable systems.

is even.[4] In addition the right hand operator products $(a^\dagger_\alpha)^s(a^\dagger_{\alpha'})^{s'}(b^\dagger_\beta)^t(b^\dagger_{\beta'})^{t'}$ must be written in the ordering given in Eqs. 5 and 6. All the above changes for fermions are duplicated for the $b^\dagger$ operator products.

The operator, $\tilde{+}$, acting on states that are linear superpositions of rational states generates entanglement. To see this Let $\psi = \sum_{\alpha,s,\beta,t} d_{\alpha,s,\beta,t}|\alpha s,\beta t\rangle$ and $\psi' = \sum_{\alpha',s',\beta',t'} d'_{\alpha',s',\beta',t'}|\alpha' s',\beta' t'\rangle$. Then

$$\begin{array}{c}\tilde{+}\psi\,\psi'|0\rangle = \sum_{\alpha,s,\beta,t}\sum_{\alpha',s',\beta',t'} d_{\alpha,s,\beta,t}d'_{\alpha',s',\beta',t'} \\ \times|\alpha s,\beta t\rangle|\alpha' s',\beta' t'\rangle|\alpha s,\beta t+\alpha' s',\beta' t'\rangle\end{array} \tag{27}$$

which is entangled.

To describe repeated arithmetic operations it is useful to have a state that describes the result of addition of ψ to ψ'. This state is obtained by taking the trace over the first two components of $\tilde{+}\psi\psi'|0\rangle$ in Eq. 27:

$$\begin{array}{l}\rho_{\psi+\psi'} = Tr_{1,2}\tilde{+}|\psi\rangle|\psi'\rangle|0\rangle\langle 0|\langle\psi'|\langle\psi|\tilde{+}^\dagger = \sum_{\alpha,\beta,s,t} \\ \times\sum_{\alpha',\beta',s',t'}|d_{\alpha,s,\beta,t}|^2|d'_{\alpha',s',\beta',t'}|^2\rho_{\alpha s,\beta t+\alpha' s',\beta' t'}.\end{array} \tag{28}$$

Here $\rho_{\alpha s,\beta t+\alpha' s',\beta' t'}$ is the pure state density operator $|\alpha s,\beta t+\alpha' s',\beta' t'\rangle\langle\alpha s,\beta t+\alpha' s',\beta' t'|$. The expectation value of $\tilde{N}$ on this state gives

$$Tr(\tilde{N}\rho_{\psi+\psi'}) = \langle\psi|\tilde{N}|\psi\rangle + \langle\psi'|\tilde{N}|\psi'\rangle \tag{29}$$

which is as expected.

For subtraction one notes that $|\alpha' s,\beta' t\rangle$ is the additive inverse of $|\alpha s,\beta t\rangle$ if $\alpha'\neq\alpha$ and $\beta'\neq\beta$. Then

$$|\alpha s,\beta t+\alpha' s,\beta' t\rangle =_N |0\rangle \tag{30}$$

where Eq. 10 is used to give $(a^\dagger_\alpha)^s(a^\dagger_{\alpha'})^s =_N 1 =_N (b^\dagger_\beta)^t(b^\dagger_{\beta'})^t$. This can be used to define a unitary subtraction operator $\tilde{-}$ by

$$\tilde{-}|\alpha s,\beta t\rangle|\alpha' s',\beta' t'\rangle|0\rangle = \tilde{+}|\alpha s,\beta t\rangle|\alpha'' s,\beta'' t\rangle|0\rangle \tag{31}$$

where $\alpha''\neq\alpha'$ and $\beta''\neq\beta'$.

5 Summary and Discussion

In this paper a quantum mechanical representation of complex rational numbers in binary was given that does not depend on qubits. Instead binary numbers are represented as a distribution of fermion or boson systems along an integer lattice that correspond to a distribution of $\pm 1s$ along the lattice. The creation operators for bosons are $a^\dagger_{\alpha,j}b^\dagger_{\beta,j}$. For fermions they are $a^\dagger_{\alpha,h,j}b^\dagger_{\beta,h,j}$. Here $\alpha,\beta = +,-$ and h is an additional parameter to allow for more than one fermion with the same sign and j value.

[4] One way to make the sign always + is to require that the dynamical steps of addition conserve fermion number by use of an additional set of fermions to serve as a sink or source.

Complex rational numbers are represented by states given as strings of creation operators acting on the vacuum state $|0\rangle$. Both standard and nonstandard representations of numbers occur. Standard representations are limited to the states with operator strings that cannot be simplified by use of Eqs. 10 and 11 for bosons, and 12 and 13 for fermions and all $a^\dagger$ operators have the same sign and all $b^\dagger$ operators have the same sign. All other basis states correspond to nonstandard representations.

This representation has some advantages over the usual qubit representation. It is compact as it avoids the use of $0s$ which serve only as place holders, and is especially useful to represent numbers with $1s$ separated by long strings of $0s$. It is also suitable for representation of columns of positive and negative real and imaginary numbers that are to be added together as one nonstandard state.

The representation used here may also suggest new physical models for quantum computers. For bosons an example would be a string of four possible types of Bose Einstein condensates (BEC) at each lattice site. The four types correspond to positive real ($a^\dagger_+$), negative real ($a^\dagger_-$), positive imaginary ($b^\dagger_+$), negative imaginary ($b^\dagger_-$). Values are determined by the numbers of each type of system at each site j. Computation operations correspond to changing the numbers of systems at the lattice sites. A similar representation for fermions is possible except that more than one fermion of the a type or of the b type at the same lattice location and with the same sign would have different h values.

Acknowledgements

This work was supported by the U.S. Department of Energy, Office of Nuclear Physics, under Contract No. W-31-109-ENG-38.

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